

Lacunary ideal convergence in probabilistic normed spaces

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ABSTRACT. An ideal I is a family of subsets of positive integers \mathbb{N} which is closed under taking finite unions and subsets of its elements. A sequence (x_k) of real numbers is said to be lacunary I -convergent to a real number ℓ , if for each $\varepsilon > 0$ the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k - \ell| \geq \varepsilon \right\}$$

belongs to I . The aim of this paper is to study the notion of lacunary I -convergence in probabilistic normed spaces as a variant of the notion of ideal convergence. Also lacunary I -limit points and lacunary I -cluster points have been defined and the relation between them has been established. Furthermore, lacunary-Cauchy and lacunary I -Cauchy sequences are introduced and studied. Finally, we provided example which shows that our method of convergence in probabilistic normed spaces is more general.

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Key words: Ideal convergence; probabilistic normed space; lacunary sequence; θ -convergence.

1. INTRODUCTION

Steinhaus [54] and Fast [17] independently introduced the notion of statistical convergence for sequences of real numbers. Over the years and under different names statistical convergence has been discussed in the theory of Fourier analysis, ergodic theory and number theory. Later on it was further investigated from various points of view. For example, statistical convergence has been investigated in summability theory by (Connor [10], Fridy [19], Šalát [49]), number theory and mathematical analysis by (Buck [3], Mitrinović et al., [46]), topological groups (Çakalli ([4, 5])), topological spaces (Di Maio and Kočinac [43]), function spaces (Caserta and Kočinac [7]), locally convex spaces (Maddox[42]), measure theory (Cheng et al., [8], Connor and Swardson [11], Miller[45]). Fridy and Orhan [20] introduced the concept of lacunary statistical convergence. Some work on lacunary statistical convergence can be found in ([4, 21, 27, 41]).

Kostyrko, et. al [35] introduced the notion of I -convergence as a generalization of statistical convergence which is based on the structure of an admissible ideal I of subset of natural numbers \mathbb{N} . Kostyrko, et. al [36] gave some of basic properties of I -convergence and dealt with extremal I -limit points. Further details on ideal convergence can be found in ([6, 14, 15, 16, 26, 28, 29, 30, 31, 32, 39, 50, 55, 57, 58]), and many others. The notion of lacunary ideal convergence of real sequences was introduced in ([9, 56]) and Hazarika ([24, 25]), was introduced the lacunary ideal convergent sequences of fuzzy real numbers and studied some properties. Debnath [13] introduced the notion lacunary ideal convergence in

intuitionistic fuzzy normed linear spaces. Recently, Yamanci and Grdal [59] introduced the notion lacunary ideal convergence in random n -normed space.

A family of subsets of \mathbb{N} , positive integers, i.e. $I \subset 2^{\mathbb{N}}$ is an ideal on \mathbb{N} if and only if

- (i) $\phi \in I$,
- (ii) $A \cup B \in I$ for each $A, B \in I$,
- (iii) each subset of an element of I is an element of I .

A non-empty family of sets $F \subset 2^{\mathbb{N}}$ is a filter on \mathbb{N} if and only if

- (a) $\phi \notin F$
- (b) $A \cap B \in F$ for each $A, B \in F$,
- (c) any subset of an element of F is in F .

An ideal I is called *non-trivial* if $I \neq \phi$ and $\mathbb{N} \notin I$. Clearly I is a non-trivial ideal if and only if $F = F(I) = \{\mathbb{N} - A : A \in I\}$ is a filter in \mathbb{N} , called the filter associated with the ideal I .

A non-trivial ideal I is called *admissible* if and only if $\{\{n\} : n \in \mathbb{N}\} \subset I$. A non-trivial ideal I is *maximal* if there cannot exist any non-trivial ideal $J \neq I$ containing I as a subset. Recall that a sequence $x = (x_k)$ of points in \mathbb{R} is said to be I -convergent to a real number ℓ if $\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\} \in I$ for every $\varepsilon > 0$ ([35]). In this case we write $I - \lim x_k = \ell$.

By a lacunary sequence $\theta = (k_r)$, where $k_0 = 0$, we shall mean an increasing sequence of non-negative integers with $k_r - k_{r-1} \rightarrow \infty$ as $r \rightarrow \infty$. The intervals determined by θ will be denoted by $J_r = (k_{r-1}, k_r]$ and we let $h_r = k_r - k_{r-1}$. The space of lacunary strongly convergent sequences \mathcal{N}_θ was defined by Freedman et al. [18] as follows:

$$\mathcal{N}_\theta = \left\{ x = (x_k) : \lim_r \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| = 0, \text{ for some } L \right\}.$$

Menger [44] proposed the probabilistic concept of the distance by replacing the number $d(p, q)$ as the distance between points p, q by a probability distribution function $F_{p,q}(x)$. He interpreted $F_{p,q}(x)$ as the probability that the distance between p and q is less than x . This led to the development of the area now called probabilistic metric spaces. This is Šerstnev [53] who first used this idea of Menger to introduce the concept of a PN space. In 1993, Alsina et al. [1] presented a new definition of probabilistic normed space which includes the definition of Šerstnev as a special case. For an extensive view on this subject, we refer ([2, 12, 22, 23, 33, 38, 40, 51, 52]). Subsequently, Mursaleen and Mohiuddine [47] and Rahmat [48] studied the ideal convergence in probabilistic normed spaces and V. Kumar and K. Kumar [37] studied I -Cauchy and I^* -Cauchy sequences in probabilistic normed spaces.

The notion of statistical convergence depends on the density (asymptotic or natural) of subsets of \mathbb{N} . A subset of \mathbb{N} is said to have natural density $\delta(E)$ if

$$\delta(E) = \lim_{n \rightarrow \infty} \frac{1}{n} |\{k \leq n : k \in E\}| \text{ exists.}$$

Definition 1.1. A sequence $x = (x_k)$ is said to be *statistically convergent* to ℓ if for every $\varepsilon > 0$

$$\delta(\{k \in \mathbb{N} : |x_k - \ell| \geq \varepsilon\}) = 0.$$

In this case, we write $S - \lim x = \ell$ or $x_k \rightarrow \ell(S)$ and S denotes the set of all statistically convergent sequences.

Definition 1.2. ([9, 56]) Let $I \subset 2^{\mathbb{N}}$ be a non-trivial ideal. A real sequence $x = (x_k)$ is said to be *lacunary I -convergent* or *I_θ -convergent* to $L \in \mathbb{R}$ if, for every $\varepsilon > 0$ the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} |x_k - L| \geq \varepsilon \right\} \in I.$$

L is called the I_θ -limit of the sequence $x = (x_k)$, and we write $I_\theta - \lim x = L$.

In this paper we study the concept of lacunary I -convergence in probabilistic normed spaces. We also define lacunary I -limit points and lacunary I -cluster points in probabilistic normed space and prove some interesting results.

2. BASIC DEFINITIONS AND NOTATIONS

Now we recall some notations and basic definitions that we are going to use in this paper.

Definition 2.1. A *distribution function* (briefly a d.f.) F is a function from the extended reals $(-\infty, +\infty)$ into $[0, 1]$ such that

- (a) it is non-decreasing ;
- (b) it is left-continuous on $(-\infty, +\infty)$;
- (c) $F(-\infty) = 0$ and $F(+\infty) = 1$.

The set of all d.f.'s will be denoted by Δ . The subset of Δ consisting of proper d.f.'s, namely of those elements F such that $\ell^+ F(-\infty) = F(-\infty) = 0$ and $\ell^- F(+\infty) = F(+\infty) = 1$ will be denoted by D . A *distance distribution function* (briefly, d.d.f.) is a d.f. F such that $F(0) = 0$. The set of all d.d.f.'s will be denoted by Δ^+ , while $D^+ := D \cap \Delta^+$ will denote the set of proper d.d.f.'s.

Definition 2.2. A *triangular norm* or, briefly, a *t -norm* is a binary operation $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ that satisfies the following conditions (see [34]):

- (T1) T is commutative, i.e., $T(s, t) = T(t, s)$ for all s and t in $[0, 1]$;
- (T2) T is associative, i.e., $T(T(s, t), u) = T(s, T(t, u))$ for all s, t and u in $[0, 1]$;
- (T3) T is nondecreasing, i.e., $T(s, t) \leq T(s', t)$ for all $t \in [0, 1]$ whenever $s \leq s'$;
- (T4) T satisfies the boundary condition $T(1, t) = t$ for every $t \in [0, 1]$.

T^* is a continuous *t -conorm*, namely, a continuous binary operation on $[0, 1]$ that is related to a continuous t -norm through $T^*(s, t) = 1 - T(1 - s, 1 - t)$. Notice that by virtue of its commutativity, any t -norm T is nondecreasing in each place. Some examples of t -norms T and its t -conorms T^* are: $M(x, y) = \min\{x, y\}$, $\Pi(x, y) = x \cdot y$ and $M^*(x, y) = \max\{x, y\}$, $\Pi^*(x, y) = x + y - x \cdot y$.

Using the definitions just given above Šerstnev [53] defined a PN space as follows:

Definition 2.3. A triplet (X, ν, T) is called a *probabilistic normed space* (in short PNS) if X is a real vector space, ν is a mapping from X into D and for $x \in X$, the d.f. $\nu(x)$ is denoted by ν_x , $\nu_x(t)$ is the value of ν_x at $t \in \mathbb{R}$ and T is a t -norm. ν satisfies the following conditions :

- (i) $\nu_x(0) = 0$;
- (ii) $\nu_x(t) = 1$ for all $t > 0$ if and only if $x = 0$;
- (iii) $\nu_{ax}(t) = \nu_x\left(\frac{t}{|a|}\right)$ for all $a \in \mathbb{R} \setminus \{0\}$;
- (iv) $\nu_{x+y}(s+t) \geq T(\nu_x(s), \nu_y(t))$ for all $x, y \in X$ and $s, t \in \mathbb{R}_0^+$.

Let $(X, \|\cdot\|)$ be a normed space and $\mu \in D$ with $\mu(0) = 0$ and $\mu \neq \epsilon_0$, where

$$\epsilon_0(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0 \end{cases}$$

For $x \in X, t \in \mathbb{R}$, if we define

$$\nu_x(t) = \mu\left(\frac{t}{\|x\|}\right), x \neq 0,$$

then in [40], it is proved that (X, ν, T) is a PN space in the sense of Definition 2.3. Alsina et al. [1] gave new definition of a PN-Space. Before giving this, we recall for the reader's convenience the concept of a triangle function, that of a PN space from the point of view of the new definition.

Definition 2.4. A triangle function is a mapping τ from $\Delta^+ \times \Delta^+$ into Δ^+ such that, for all F, G, H, K in Δ^+ ,

- (1) $\tau(F, \epsilon_0) = F$;
- (2) $\tau(F, G) = \tau(G, F)$;
- (3) $\tau(F, G) \leq \tau(H, K)$ whenever $F \leq H, G \leq K$;
- (4) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

Particular and relevant triangle functions are the functions τ_T, τ_{T^*} and those of the form Π_T which, for any continuous t -norm T , and any $x > 0$, are given by

$$\tau_T(F, G)(x) = \sup\{T(F(u), G(v)) : u + v = x\},$$

$$\tau_{T^*}(F, G)(x) = \inf\{T^*(F(u), G(v)) : u + v = x\}$$

and

$$\Pi_T(F, G)(x) = T(F(x), G(x)).$$

Definition 2.5. ([1]) A probabilistic normed space is a quad-ruple (X, ν, τ, τ^*) , where X is a real linear space, τ and τ^* are continuous triangle functions such that $\tau \leq \tau^*$ and the mapping $\nu : X \rightarrow \Delta^+$ called the probabilistic norm, satisfies for all p and q in X , the conditions

- (PN1) $\nu_p = \epsilon_0$ if and only if $p = \theta$ (θ is the null vector in X);
- (PN2) $\forall p \in X, \nu_{-p} = \nu_p$;
- (PN3) $\nu_{p+q} \geq \tau(\nu_p, \nu_q)$;
- (PN4) $\forall a \in [0, 1], \nu_p \leq \tau^*(\nu_{ap}, \nu_{(1-a)p})$.

If a PN space (X, ν, τ, τ^*) , satisfies the following condition

(Š) $\forall p \in X, \forall \lambda \in \mathbb{R} \setminus \{0\}, \forall t > 0, \nu_{\lambda p}(t) = \nu_p\left(\frac{t}{|\lambda|}\right)$, then it is called a Šherstnev PN

space; the condition (Š) implies that the best-possible selection for τ^* is $\tau^* = \tau_M$, which satisfies a stricter version of (PN4), namely,

$$\forall a \in [0, 1], \nu_p = \tau_M(\nu_{ap}, \nu_{(1-a)p}).$$

Definition 2.6. A Menger PN space under T is a PN space (X, ν, τ, τ^*) denoted by (X, ν, T) , in which $\tau = \tau_T$ and $\tau^* = \tau_{T^*}$, for some continuous t -norm T and its t -conorm T^* .

Lemma 2.1 ([40]) *The simple space generated by $(X, \|\cdot\|)$ and by μ is a Menger PN space under M and also a Šherstnev PN space. Here $M(x, y) := \min\{x, y\}$.*

For further study, by a PN space we mean a PN space in the sense of Definition 2.3. We now give a quick look on the characterization of convergence and Cauchy sequences on these spaces.

Let (X, ν, T) be a PN space and $x = (x_k)$ be a sequence in X . We say that (x_k) is *convergent* to $\ell \in X$ with respect to the probabilistic norm ν if for each $\varepsilon > 0$ and $\alpha \in (0, 1)$ there exists a positive integer m such that $\nu_{x_k - \ell}(\varepsilon) > 1 - \alpha$ whenever $k \geq m$. The element ℓ is called the ordinary double limit of the sequence (x_k) and we shall write $\nu - \lim x_k = \ell$ or $x_k \xrightarrow{\nu} \ell$ as $k \rightarrow \infty$.

A sequence (x_k) in X is said to be *Cauchy* with respect to the probabilistic norm ν if for each $\varepsilon > 0$ and $\alpha \in (0, 1)$ there exist a positive integer $M = M(\varepsilon, \alpha)$ such that $\nu_{x_k - x_p}(\varepsilon) > 1 - \alpha$ whenever $k, p \geq M$.

Definition 2.7. Let (X, ν, T) be an probabilistic normed space, and let $r \in (0, 1)$ and $x \in X$. The set

$$B(x, r; t) = \{y \in X : \nu_{y-x}(t) > 1 - r\}$$

is called open ball with center x and radius r with respect to t .

Throughout the paper, we denote I is an admissible ideal of subsets of \mathbb{N} and $\theta = (k_r)$, respectively, unless otherwise stated.

3. MAIN RESULTS

We now obtain our main results.

Definition 3.1. Let $I \subset 2^{\mathbb{N}}$ and (X, ν, T) be an PNS. A sequence $x = (x_k)$ in X is said to be I_θ -convergent to $L \in X$ with respect to the probabilistic norm ν if, for every $\varepsilon > 0$ and $\alpha \in (0, 1)$ the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

L is called the I_θ -limit of the sequence $x = (x_k)$ in X , and we write $I_\theta^\nu - \lim x = L$.

Example 3.1. Let $(\mathbb{R}, |\cdot|)$ denote the space of all real numbers with the usual norm, and let $T(a, b) = ab$ for all $a, b \in [0, 1]$. For all $x \in \mathbb{R}$ and every $t > 0$, consider $\nu_x(t) = \frac{t}{t+|x|}$. Then (\mathbb{R}, ν, T) is an PNS. If we take $I = \{A \subset \mathbb{N} : \delta(A) = 0\}$, where $\delta(A)$ denotes the natural density of the set A , then I is a non-trivial admissible ideal. Define a sequence $x = (x_k)$ as follows:

$$x_k = \begin{cases} 1, & \text{if } k = i^2, i \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Then for every $\alpha \in (0, 1)$ and for any $\varepsilon > 0$, the set

$$K = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k}(\varepsilon) \leq 1 - \alpha \right\}$$

will be a finite set. Hence, $\delta(K) = 0$ and consequently $K \in I$, i.e., $I_\theta^\nu - \lim x = 0$.

Lemma 3.1. Let (X, ν, T) be an PNS and $x = (x_k)$ be a sequence in X . Then, for every $\varepsilon > 0$ and $\alpha \in (0, 1)$ the following statements are equivalent:

- (i) $I_\theta^\nu - \lim x = L$,
- (ii) $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \in I$
- (iii) $\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha \right\} \in F(I)$,
- (iii) $I_\theta - \lim \nu_{x_k - L}(\varepsilon) = 1$.

Theorem 3.1. *Let (X, ν, T) be an PNS and if a sequence $x = (x_k)$ in X is I_θ -convergent to $L \in X$ with respect to the probabilistic norm ν , then $I_\theta^\nu - \lim x$ is unique.*

Proof. Suppose that $I_\theta^\nu - \lim x = L_1$ and $I_\theta^\nu - \lim x = L_2$ ($L_1 \neq L_2$). Given $\alpha > 0$ and choose $\beta \in (0, 1)$ such that

$$(3.1) \quad T(1 - \beta, 1 - \beta) > 1 - \alpha.$$

Then for $\varepsilon > 0$, define the following sets:

$$K_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1} \left(\frac{\varepsilon}{2} \right) \leq 1 - \beta \right\},$$

$$K_2 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2} \left(\frac{\varepsilon}{2} \right) \leq 1 - \beta \right\},$$

Since $I_\theta^\nu - \lim x = L_1$, using Lemma 2.1., we have $K_1 \in I$. Also, using $I_\theta^\nu - \lim x = L_2$, we get $K_2 \in I$. Now let

$$K = K_1 \cup K_2.$$

Then $K \in I$. This implies that its complement K^c is a non-empty set in $F(I)$. Now if $r \in K^c$, let us consider $r \in K_1^c \cap K_2^c$. Then we have

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1} \left(\frac{\varepsilon}{2} \right) > 1 - \beta \text{ and } \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2} \left(\frac{\varepsilon}{2} \right) > 1 - \beta.$$

Now, we choose a $s \in \mathbb{N}$ such that

$$\nu_{x_s - L_1} \left(\frac{\varepsilon}{2} \right) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1} \left(\frac{\varepsilon}{2} \right) > 1 - \beta$$

and

$$\nu_{x_s - L_2} \left(\frac{\varepsilon}{2} \right) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2} \left(\frac{\varepsilon}{2} \right) > 1 - \beta$$

e.g., consider $\max \left\{ \nu_{x_k - L_1} \left(\frac{\varepsilon}{2} \right), \nu_{x_k - L_2} \left(\frac{\varepsilon}{2} \right) : k \in J_r \right\}$ and choose that k as s for which the maximum occurs. Then from (2.1), we have

$$\nu_{L_1 - L_2}(\varepsilon) \geq T \left(\nu_{x_s - L_1} \left(\frac{\varepsilon}{2} \right), \nu_{x_s - L_2} \left(\frac{\varepsilon}{2} \right) \right) > T(1 - \beta, 1 - \beta) > 1 - \alpha.$$

Since $\alpha > 0$ is arbitrary, we have $\nu_{L_1 - L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which implies that $L_1 = L_2$. Therefore, we conclude that $I_\theta^\nu - \lim x$ is unique.

Here, we introduce the notion of θ -convergence in an PNS and discuss some properties.

Definition 3.2. Let (X, ν, T) be an PNS. A sequence $x = (x_k)$ in X is θ -convergent to $L \in X$ with respect to the probabilistic norm ν if, for $\alpha \in (0, 1)$ and every $\varepsilon > 0$, there exists $r_o \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_o$. In this case, we write $\nu^\theta - \lim x = L$.

Theorem 3.2. *Let (X, ν, T) be an PNS and let $x = (x_k)$ in X . If $x = (x_k)$ is θ -convergent with respect to the probabilistic norm ν , then $\nu^\theta - \lim x$ is unique.*

Proof. Suppose that $\nu^\theta - \lim x = L_1$ and $\nu^\theta - \lim x = L_2$ ($L_1 \neq L_2$). Given $\alpha \in (0, 1)$ and choose $\beta \in (0, 1)$ such that $T(1 - \beta, 1 - \beta) > 1 - \alpha$. Then for any $\varepsilon > 0$, there exists $r_1 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_1$. Also, there exists $r_2 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_2$. Now, consider $r_o = \max\{r_1, r_2\}$. Then for $r \geq r_o$, we will get a $s \in \mathbb{N}$ such that

$$\nu_{x_s - L_1}\left(\frac{\varepsilon}{2}\right) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1}\left(\frac{\varepsilon}{2}\right) > 1 - \beta$$

and

$$\nu_{x_s - L_2}\left(\frac{\varepsilon}{2}\right) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_2}\left(\frac{\varepsilon}{2}\right) > 1 - \beta.$$

Then, we have

$$\nu_{L_1 - L_2}(\varepsilon) \geq T\left(\nu_{x_s - L_1}\left(\frac{\varepsilon}{2}\right), \nu_{x_s - L_2}\left(\frac{\varepsilon}{2}\right)\right) > T(1 - \beta, 1 - \beta) > 1 - \alpha.$$

Since $\alpha > 0$ is arbitrary, we have $\nu_{L_1 - L_2}(\varepsilon) = 1$ for all $\varepsilon > 0$, which implies that $L_1 = L_2$.

Theorem 3.3. Let (X, ν, T) be an PNS and let $x = (x_k)$ in X . If $\nu^\theta - \lim x = L$, then $I_\theta^\nu - \lim x = L$.

Proof. Let $\nu^\theta - \lim x = L$, then for every $\varepsilon > 0$ and given $\alpha \in (0, 1)$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_0$. Therefore the set

$$B = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\} \subseteq \{1, 2, \dots, n_0 - 1\}.$$

But, with I being admissible, we have $B \in I$. Hence $I_\theta^\nu - \lim x = L$.

Theorem 3.4. Let (X, ν, T) be an PNS and $x = (x_k), y = (y_k)$ be two sequence in X .

(i) If $I_\theta^\nu - \lim x_k = L_1$ and $I_\theta^\nu - \lim y_k = L_2$, then $I_\theta^\nu - \lim(x_k \pm y_k) = L_1 \pm L_2$;

(ii) If $I_\theta^\nu - \lim x_k = L$ and a be a non-zero real number, then $I_\theta^\nu - \lim ax_k = aL$. If $a = 0$, then result is true only if I is an admissible of \mathbb{N} .

Proof. (i) We have proved that, if $I_\theta^\nu - \lim x_k = L_1$ and $I_\theta^\nu - \lim y_k = L_2$, then $I_\theta^\nu - \lim(x_k + y_k) = L_1 + L_2$, only. The proof of the other part follows similarly.

Take $\varepsilon > 0, \alpha \in (0, 1)$ and choose $\beta \in (0, 1)$ such that the condition (3.1) holds. If we define

$$A_1 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_1}\left(\frac{\varepsilon}{2}\right) \leq 1 - \beta \right\},$$

and

$$A_2 = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L_2} \left(\frac{\varepsilon}{2} \right) \leq 1 - \beta \right\}.$$

Then $A_1^c \cap A_2^c \in F(I)$. We claim that

$$A_1^c \cap A_2^c \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) > 1 - \alpha \right\}.$$

Let $n \in A_1^c \cap A_2^c$. Now, using (3.1), we have

$$\begin{aligned} \frac{1}{h_r} \sum_{n \in J_r} \nu_{(x_n - L_1) + (y_n - L_2)}(\varepsilon) &\geq T \left(\frac{1}{h_r} \sum_{n \in J_r} \nu_{x_n - L_1} \left(\frac{\varepsilon}{2} \right), \frac{1}{h_r} \sum_{n \in J_r} \nu_{y_n - L_2} \left(\frac{\varepsilon}{2} \right) \right) \\ &> T(1 - \beta, 1 - \beta) > 1 - \alpha. \end{aligned}$$

Hence

$$A_1^c \cap A_2^c \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) > 1 - \alpha \right\}.$$

As $A_1^c \cap A_2^c \in F(I)$, so

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{(x_k - L_1) + (y_k - L_2)}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

Therefore $I_\theta^\nu - \lim(x_k + y_k) = L_1 + L_2$.

(ii) Suppose $a \neq 0$. Since $I_\theta^\nu - \lim x_k = L$, for each $\varepsilon > 0$ and $\alpha \in (0, 1)$, the set

$$A(\varepsilon, \alpha) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) < 1 - \alpha \right\} \in F(I).$$

If $n \in A(\varepsilon, \alpha)$, then we have

$$\begin{aligned} \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) &= \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L} \left(\frac{\varepsilon}{|a|} \right) \\ &\geq T \left(\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon), \nu_0 \left(\frac{\varepsilon}{|a|} - \varepsilon \right) \right) \\ &\geq T \left(\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon), 1 \right) \geq \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha \end{aligned}$$

Hence

$$A(\varepsilon, \alpha) \subset \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) > 1 - \alpha \right\}$$

and

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) > 1 - \alpha \right\} \in F(I).$$

It follows that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{ax_k - aL}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

Hence $I_\theta^\nu - \lim ax_k = aL$.

Next suppose that $a = 0$. Then for each $\varepsilon > 0$ and $\alpha \in (0, 1)$, we have

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{0x_k - 0L}(\varepsilon) = \frac{1}{h_r} \sum_{k \in J_r} \nu_0(\varepsilon) = 1 > 1 - \alpha,$$

it follows that $\nu^\theta - \lim x = \ell$. Hence from Theorem 3.3, $I_\theta^\nu - \lim x = \ell$.

Theorem 3.5. *Let (X, ν, T) be an PNS and let $x = (x_k)$ in X . If $\nu^\theta - \lim x = L$, then there exists a subsequence (x_{m_k}) of $x = (x_k)$ such that $\nu - \lim x_{m_k} = L$.*

Proof. Let $\nu^\theta - \lim x = L$. Then, for every $\varepsilon > 0$ and given $\alpha \in (0, 1)$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_0$. Clearly, for each $r \geq r_0$, we can select an $m_k \in J_r$ such that

$$\nu_{x_{m_k} - L}(\varepsilon) > \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha.$$

It follows that $\nu - \lim x_{m_k} = L$.

Definition 3.3. Let (X, ν, T) be an PNS and let $x = (x_k)$ be a sequence in X . Then,

- (1) An element $L \in X$ is said to be I_θ -limit point of $x = (x_k)$ if there is a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that the set $M^c = \{r \in \mathbb{N} : m_k \in J_r\} \notin I$ and $\nu^\theta - \lim x_{m_k} = L$.
- (2) An element $L \in X$ is said to be I_θ -cluster point of $x = (x_k)$ if for every $\varepsilon > 0$ and $\alpha \in (0, 1)$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha \right\} \notin I.$$

Let $\Lambda_\nu^{I_\theta}(x)$ denote the set of all I_θ -limit points and $\Gamma_\nu^{I_\theta}(x)$ denote the set of all I_θ -cluster points in X , respectively.

Theorem 3.6. *Let (X, ν, T) be an PNS. For each sequence $x = (x_k)$ in X , we have $\Lambda_\nu^{I_\theta}(x) \subset \Gamma_\nu^{I_\theta}(x)$.*

Proof. Let $L \in \Lambda_\nu^{I_\theta}(x)$, then there exists a set $M \subset \mathbb{N}$ such that $M^c \notin I$, where M and M^c are as in the Definition 2.3., satisfies $\nu^\theta - \lim x_{m_k} = L$. Thus, for every $\varepsilon > 0$ and $\alpha \in (0, 1)$, there exists $r_0 \in \mathbb{N}$ such that

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_{m_k} - L}(\varepsilon) > 1 - \alpha$$

for all $r \geq r_0$. Therefore,

$$B = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) > 1 - \alpha \right\} \supseteq M^c \setminus \{m_1, m_2, \dots, m_{n_0}\}.$$

Now, with I being admissible, we must have $M^c \setminus \{m_1, m_2, \dots, m_{n_0}\} \notin I$ and as such $B \notin I$. Hence $L \in \Gamma_\nu^{I_\theta}(x)$.

Theorem 3.7. *Let (X, ν, T) be an PNS. For each sequence $x = (x_k)$ in X , the set $\Gamma_\nu^{I_\theta}(x)$ is closed set in X with respect to the usual topology induced by the probabilistic norm ν^θ .*

Proof. Let $y \in \overline{\Gamma_\nu^{I_\theta}(x)}$. Take $\varepsilon > 0$ and $\alpha \in (0, 1)$. Then there exists $L_0 \in \Gamma_\nu^{I_\theta}(x) \cap B(y, \alpha, \varepsilon)$. Choose $\delta > 0$ such that $B(L_0, \delta, \varepsilon) \subset B(y, \alpha, \varepsilon)$. We have

$$\begin{aligned} G &= \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - y}(\varepsilon) > 1 - \alpha \right\} \\ &\supseteq \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L_0}(\varepsilon) > 1 - \delta \right\} = H. \end{aligned}$$

Thus $H \notin I$ and so $G \notin I$. Hence $y \in \Gamma_\nu^{I_\theta}(x)$.

Theorem 3.8. Let (X, ν, T) be an PNS and let $x = (x_k)$ in X . Then the following statements are equivalent:

- (1) L is a I_θ -limit point of x ,
- (2) There exist two sequences y and z in X such that $x = y + z$ and $\nu^\theta - \lim y = L$ and $\{r \in \mathbb{N} : k \in J_r, z_k \neq \bar{\theta}\} \in I$, where $\bar{\theta}$ is the zero element of X .

Proof. Suppose that (1) holds. Then there exist sets M and M^c as in Definition 2.3. such that $M^c \notin I$ and $\nu^\theta - \lim x_{m_k} = L$. Define the sequences y and z as follows:

$$y_k = \begin{cases} x_k, & \text{if } k \in J_r; r \in M^c, \\ L, & \text{otherwise.} \end{cases}$$

and

$$z_k = \begin{cases} \bar{\theta}, & \text{if } k \in J_r; r \in M^c, \\ x_k - L, & \text{otherwise.} \end{cases}$$

It suffices to consider the case $k \in J_r$ such that $r \in \mathbb{N} \setminus M^c$. Then for each $\alpha \in (0, 1)$ and $\varepsilon > 0$, we have $\nu_{y_k - L}(\varepsilon) = 1 > 1 - \alpha$. Thus, in this case,

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L}(\varepsilon) = 1 > 1 - \alpha.$$

Hence $\nu^\theta - \lim y = L$. Now $\{r \in \mathbb{N} : k \in J_r, z_k \neq \bar{\theta}\} \subset \mathbb{N} \setminus M^c$ and so $\{r \in \mathbb{N} : k \in J_r, z_k \neq \bar{\theta}\} \in I$.

Now, suppose that (2) holds. Let $M^c = \{r \in \mathbb{N} : k \in J_r, z_k = \bar{\theta}\}$. Then, clearly $M^c \in F(I)$ and so it is an infinite set. Construct the set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that $m_k \in J_r$ and $z_{m_k} = \bar{\theta}$. Since $x_{m_k} = y_{m_k}$ and $\nu^\theta - \lim y = L$ we obtain $\nu^\theta - \lim x_{m_k} = L$. This completes the proof.

Theorem 3.9. Let (X, ν, T) be an PNS and $x = (x_k)$ be a sequence in X . Let I be an admissible ideal in \mathbb{N} . If there is a I_θ^ν -convergent sequence $y = (y_k)$ in X such that $\{k \in \mathbb{N} : y_k \neq x_k\} \in I$ then x is also I_θ^ν -convergent.

Proof. Suppose that $\{k \in \mathbb{N} : y_k \neq x_k\} \in I$ and $I_\theta^\nu - \lim y = \ell$. Then for every $\alpha \in (0, 1)$ and $\varepsilon > 0$, the set

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L}(\varepsilon) \leq 1 - \alpha \right\} \in I.$$

For every $0 < \alpha < 1$ and $\varepsilon > 0$, we have

$$(3.2) \quad \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - L}(\varepsilon) \leq 1 - \alpha \right\}$$

$$\subseteq \{k \in \mathbb{N} : y_k \neq x_k\} \cup \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{y_k - L}(\varepsilon) \leq 1 - \alpha \right\}.$$

As both the sets of right-hand side of (2.2) is in I , therefore we have that

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \mu(x_k - L, t) \leq 1 - \varepsilon \text{ or } \frac{1}{h_r} \sum_{k \in J_r} \nu(x_k - L, t) \geq \varepsilon \right\} \in I.$$

This completes the proof of the theorem.

Definition 3.4. Let (X, ν, T) be an PNS. A sequence $x = (x_k)$ in X is said to be θ -Cauchy sequence with respect to the probabilistic norm ν if, for every $\varepsilon > 0$ and $\alpha \in (0, 1)$, there exist $r_0, m \in \mathbb{N}$ satisfying

$$\frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - x_m}(\varepsilon) > 1 - \varepsilon$$

for all $r \geq r_0$.

Definition 3.5. Let I be an admissible ideal of \mathbb{N} . Let (X, ν, T) be an PNS. A sequence $x = (x_k)$ in X is said to be I_θ -Cauchy sequence with respect to the probabilistic norm ν if, for every $\varepsilon > 0$ and $\alpha \in (0, 1)$, there exists $m \in \mathbb{N}$ satisfying

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in J_r} \nu_{x_k - x_m}(\varepsilon) > 1 - \varepsilon \right\} \in F(I)$$

Definition 3.6. Let I be an admissible ideal of \mathbb{N} . Let (X, ν, T) be an PNS. A sequence $x = (x_k)$ in X is said to be I_θ^* -Cauchy sequence with respect to the probabilistic norm ν if, there exists a set $M = \{m_1 < m_2 < \dots < m_k < \dots\} \subset \mathbb{N}$ such that the set $M^c = \{r \in \mathbb{N} : m_k \in J_r\} \in F(I)$ and the subsequence (x_{m_k}) of $x = (x_k)$ is a θ -Cauchy sequence with respect to the probabilistic norm ν .

The following theorem is an analogue of Theorem 3.3, so the proof omitted.

Theorem 3.10. *Let I be an admissible ideal of \mathbb{N} . Let (X, ν, T) be an PNS. If a sequence $x = (x_k)$ in X is θ -Cauchy sequence with respect to the probabilistic norm ν , then it is I_θ -Cauchy sequence with respect to the same norm.*

The proof of the following theorem's proof is similar to that of Theorem 3.5.

Theorem 3.11. *Let (X, ν, T) be an PNS. If a sequence $x = (x_k)$ in X is θ -Cauchy sequence with respect to the probabilistic norm ν , then there is a subsequence of $x = (x_k)$ which is ordinary Cauchy sequence with respect to the same norm.*

The following theorem can be proved easily using similar techniques as in the proof of Theorem 3.6.

Theorem 3.12. *Let I be an admissible ideal of \mathbb{N} . Let (X, ν, T) be an PNS. If a sequence $x = (x_k)$ in X is I_θ^* -Cauchy sequence with respect to the probabilistic norm ν , then it is I_θ -Cauchy sequence as well.*

REFERENCES

- [1] C. Alsina, B. Schweizer, A. Sklar, On the definition of a probabilistic normed space. Aequationes Math., 46, 91-98(1993)

- [2] C. Alsina, B. Schweizer, A. Sklar, Continuity properties of probabilistic norms, *J. Math. Anal. Appl.* 208 (1997), 446-452.
- [3] R.C. Buck, The measure theoretic approach to density, *Amer. J. Math.* 68 (1946) 560-580.
- [4] H. Çakalli, On statistical convergence in topological groups, *Pure Appl. Math. Sci.*, 43(1996), 27-31.
- [5] H. Çakalli, A study on statistical convergence, *Funct. Anal. Approx. Comput.*, 1(2)(2009), 19-24, MR2662887.
- [6] H.Çakalli, and B. Hazarika, Ideal-quasi-Cauchy sequences, *Jour. Ineq. Appl.*, 2012(2012) pages 11, doi:10.1186/1029-242X-2012-234
- [7] A. Caserta, G. Di Maio, Lj. D. R. Kočinac, Statistical convergence in function spaces, *Abstr. Appl. Anal.* Vol. 2011(2011), Article ID 420419, 11 pages.
- [8] L. X. Cheng, G. C. Lin, Y. Y. Lan, H. Liu, Measure theory of statistical convergence, *Science in China, Ser. A: Math.* 51(2008), 2285-2303.
- [9] B. Choudhary, Lacunary I -convergent sequences, in: *Real Analysis Exchange Summer Symposium, 2009*, pp. 56-57.
- [10] J. Connor, The statistical and strong p -Cesàro convergence of sequences, *Analysis* 8 (1988) 47-63.
- [11] J. Connor, M.A. Swardson, Measures and ideals of $C^*(X)$, *Ann. N.Y. Acad.Sci.* 704(1993), 80-91.
- [12] G. Constantin, I. Istratescu, *Elements of Probabilistic Analysis*, Kluwer, 1989
- [13] P. Debnath, Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces, *Comput. Math. Appl.*, 63(2012), 708-715.
- [14] K. Dems, On I -Cauchy sequences. *Real Anal. Exchange*, 30(1)(2004/2005), 123-128
- [15] A. Esi, B. Hazarika, λ -ideal convergence in intuitionistic fuzzy 2-normed linear space, *Jour. Intell. Fuzzy Systems*, 24(4)(2013), 725-732, DOI: 10.3233/IFS-2012-0592
- [16] A. Esi, B. Hazarika, Lacunary Summable Sequence Spaces of Fuzzy Numbers Defined By Ideal Convergence and an Orlicz Function, *Afrika Matematika*, DOI: 10.1007/s13370-012-0117-3
- [17] H. Fast, Sur la convergence statistique, *Colloq. Math.* 2(1951) 241-244.
- [18] A. R. Freedman, J. J. Sember, M. Raphael, Some Cesaro-type summability spaces, *Proc. London Math. Soc.*, 37(3) (1978) 508-520.
- [19] J. A. Fridy, On statistical convergence, *Analysis*, 5(1985) 301-313.
- [20] J. A. Fridy, C. Orhan, Lacunary statistical convergence, *Pacific J. Math.*, 160(1)(1993), 43-51, MR 94j:40014.
- [21] J. A. Fridy, C. Orhan, Lacunary statistical summability, *J. Math. Anal. Appl.*, 173, (1993), 497-504, MR 95f :40004.
- [22] B.L. Guillén, J.A.R. Lallena, C. Sempì, A study of boundedness in probabilistic normed spaces. *J. Math. Anal. Appl.* 232 (1999), 183-196.
- [23] B.L. Guillén, C. Sempì, Probabilistic norms and convergence of random variables. *J. Math. Anal. Appl.* 280 (2003), 9-16.
- [24] B. Hazarika, Lacunary I -convergent sequence of fuzzy real numbers, *The Pacific Jour. Sci. Techno.*, 10(2) (2009), 203-206.
- [25] B. Hazarika, Fuzzy real valued lacunary I -convergent sequences, *Appl. Math. Letters* 25 (2012) 466470.
- [26] B. Hazarika, E. Savas, Some I -convergent lambda-summable difference sequence spaces of fuzzy real numbers defined by a sequence of Orlicz functions, *Math. Comp. Modell.* 54(2011) 2986-2998.
- [27] B. Hazarika, E. Savas, Lacunary statistical convergence of double sequences and some inclusion results in n -normed spaces, *Acta Mathematica Vietnamica*, (Accepted for publications).
- [28] B. Hazarika, Lacunary difference ideal convergent sequence spaces of fuzzy numbers, *Journal of Intelligent and Fuzzy Systems*, DOI: 10.3233/IFS-2012-0622.
- [29] B. Hazarika, On generalized difference ideal convergence in random 2-normed spaces, *Filomat*, 26(6) (2012), 1265-1274.
- [30] B. Hazarika, On σ -uniform density and ideal convergent sequences of fuzzy real numbers, *Journal of Intelligent and Fuzzy Systems*, doi 10.3233/IFS-130769.
- [31] B. Hazarika, V. Kumar, B. L. Guillén, Generalized ideal convergence in intuitionistic fuzzy normed linear spaces, *Filomat* (Accepted for publications).
- [32] B. Hazarika, On ideal convergence in topological groups, *Scientia Magna*, 7(4)(2011), 80-86.
- [33] S. Karakus, Statistical convergence on probabilistic normed spaces. *Math. Comm.* 12 (2007), 11-23.
- [34] E. P. Klement, R. Mesiar, E. Pap., *Triangular Norms*, Kluwer, Dordrecht, 2000
- [35] P. Kostyrko, T. Šalát, and W. Wilczyński, I -convergence, *Real Anal. Exchange* 26, 2, (2000-2001), 669-686, MR 2002e:54002.
- [36] P. Kostyrko, M. Macaj, T. Šalát, M. Slezia, I -convergence and Extremal I -limit points, *Math. Slovaca* 2005; 55; 443-64.
- [37] K. Kumar, V. Kumar, On the I and I^* -Cauchy sequences in probabilistic normed spaces. *Mathematical Sciences*, 2(1), 47-58 (2008)
- [38] V. Kumar, B. L. Guillén, On Ideal Convergence of Double Sequences in Probabilistic Normed Spaces, *Acta Math. Sinica, English Series*, Published online: February 21, 2012, DOI: 10.1007/s10114-012-9321-1

- [39] B. K. Lahiri, P. Das, I and I^* -convergence in topological spaces, *Math. Bohemica*, 130 (2005), 153-160.
- [40] B. Lafuerza-Guillén, J. A. Rodríguez-Lallena, C. Sempí, Some classes of probabilistic normed spaces. *Rend. Mat.*, 17(1997), 237-252
- [41] J. Li, Lacunary statistical convergence and inclusion properties between lacunary methods, *Internat. J. Math. Math. Sci.* 23(3) (2000), 175-180, S0161171200001964.
- [42] I. J. Maddox, Statistical convergence in a locally convex spaces, *Math. Proc. Cambridge Philos. Soc.*, 104(1)(1988), 141-145.
- [43] G. Di. Maio, Lj.D.R. Kočinac, Statistical convergence in topology, *Topology Appl.* 156, (2008), 28-45.
- [44] K. Menger, Statistical metrics. *Proc. Nat. Acad. Sci. USA*, 28(1942) 535-537
- [45] H. I. Miller, A measure theoretical subsequence characterization of statistical convergence, *Trans. Amer. Math. Soc.*, 347(5)(1995), 1811-1819.
- [46] D.S. Mitrinović, J. Sandor, B. Crstici, *Handbook of Number Theory*, Kluwer Acad. Publ., Dordrecht, Boston, London, 1996.
- [47] M. Mursaleen, S. A. Mohiuddine, On ideal convergence in probabilistic normed spaces, *Math. Slovaca*, 62(1)(2012), 49-62.
- [48] M. R. S. Rahmat, Ideal Convergence on Probabilistic Normed Spaces, *Inter. Jour. Stat. Econ.*, 3(9)(2009), 67-75
- [49] T. Šalát, On statistical convergence of real numbers, *Math. Slovaca*, 30(1980), 139-150.
- [50] T. Šalát, B. C. Tripathy, M. Ziman, On some properties of I -convergence, *Tatra Mt. Math. Publ.* 2004; 28; 279-86.
- [51] B. Schweizer, A. Sklar, Statistical metric spaces, *Pacific J. Math.* 10 (1960), 313-334.
- [52] B. Schweizer, A. Sklar, *Probabilistic Metric Spaces*, North Holland, New York- Amsterdam-Oxford, 1983.
- [53] A. N. Šerstnev, Random normed spaces: Problems of completeness. *Kazan Gos. Univ. Ucen. Zap.*, 122, 3-20 (1962)
- [54] H. Steinhaus, Sur la convergence ordinaire et la convergence asymptotique, *Colloq. Math.* 2 (1951) 73-74.
- [55] B. C. Tripathy, B. Hazarika, I -monotonic and I -convergent sequences, *Kyungpook Math. J.* 51(2011), 233-239, DOI 10.5666/KMJ.2011.51.2.233.
- [56] B. C. Tripathy, B. Hazarika, B. Choudhary, Lacunary I -convergent sequences, *Kyungpook Math. J.* 52(4)(2012) 473-482.
- [57] B. C. Tripathy and B. Hazarika, Paranorm I -convergent sequence spaces, *Math. Slovaca*, **59(4)** (2009) 485-494.
- [58] B. C. Tripathy, B. Hazarika, Some I -convergent sequence spaces defined by Orlicz functions, *Acta Math. Appl. Sinica*, **27(1)** (2011) 149-154.
- [59] U. Yamancı, M. Gürdal, On lacunary ideal convergence in random n -normed space, *Journal of Mathematics*, Vol. 2013(2013), Article ID 868457, 8 pages .